BERRY-ESSEEN BOUNDS
IN THE LOCAL LIMIT THEOREMS

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Abstract. Berry-Esseen-type bounds are developed in the multidimensional local limit theorem in terms of the Lyapunov coefficients and maxima of involved densities.

1. **Introduction**

Consider the normalized sum

$$Z_n = \frac{X_1 + \cdots + X_n}{\sqrt{n}}$$

of n independent random vectors X_k in $\mathbb{R}^d$ with mean zero and covariance matrices $\sigma_k^2 I_d$ ($\sigma_k > 0$) such that $\sigma_1^2 + \cdots + \sigma_n^2 = n$. The latter ensures that Z_n has mean zero and unit covariance matrix I_d.

Introduce the Lyapunov ratios of the third and fourth orders

$$\beta_3 = \sup_{|\theta|=1} \left[\frac{1}{n}\sum_{k=1}^n \mathbb{E}\,|\,\langle \theta, X_k\rangle\,|^3\right], \quad \beta_4 = \sup_{|\theta|=1} \left[\frac{1}{n}\sum_{k=1}^n \mathbb{E}\,\langle \theta, X_k\rangle^4\right],$$

assuming respectively that $\mathbb{E}\,|X_k|^3 < \infty$ and $\mathbb{E}\,|X_k|^4 < \infty$ for all $k \le n$. In dimension one, the quantity β_3 is commonly used to quantify the normal approximation for the distribution of Z_n in a weak sense via the Berry-Esseen bound

$$\sup_{x\in\mathbb{R}} \big|\mathbb{P}\{Z_n \le x\} - \mathbb{P}\{Z \le x\}\big| \le \frac{C}{\sqrt{n}}\,\beta_3,$$

where Z is a standard normal random variable and C is an absolute constant (cf. e.g. [19]). The Lyapunov ratio β_4 also appears in a natural way, for example, for the approximation of the characteristic function of Z_n by a corrected normal characteristic function. Both quantities may also be used to control the distance between the distribution of Z_n and Z on the real line in total variation and in relative entropy, cf. [9].

The aim of this note is to quantify the normal approximation in a stronger sense towards a uniform convergence of densities p_n of Z_n to the standard normal density

$$\varphi(x) = \frac{1}{(2\pi)^{d/2}}\,e^{-|x|^2/2}, \quad x \in \mathbb{R}^d.$$

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In the i.i.d. situation (when all X_k are identically distributed with $\sigma_k = 1$), the necessary and sufficient condition for the convergence of the uniform distance

$$\Delta_n = \sup_x |p_n(x) - \varphi(x)|$$

to zero as $n \to \infty$ is that p_n is bounded for some $n = n_0$ (cf. [19, 5]). As typically $n_0 = 1$ in applications, it is natural to consider the case where all X_k have bounded densities.

So, introduce the maximum-of-density functional $M(X) = \operatorname{ess\,sup}_x p(x)$, where p denotes a density of a random vector X, and define

$$M = \max_k M(X_k), \quad \sigma^2 = \max_k \sigma_k^2 \ \ (\sigma > 0).$$

The functionals β_3, β_4, σ and M can be used to derive the following upper bounds (which seem to be new already in the one dimensional situation).

Theorem 1.1. *With some positive absolute constant C, the density p_n of Z_n satisfies*

$$\Delta_n \leq \frac{(C\sigma)^d M^2}{\sqrt{n}}\, \beta_3. \tag{1.1}$$

Moreover, if $\mathbb{E}\, \langle \theta, X_k \rangle^3 = 0$ for all $\theta \in \mathbb{R}^d$ and $k \leq n$ (in particular, if the distributions of X_k are symmetric), then

$$\Delta_n \leq \frac{(C\sigma)^{2d} M^3}{n}\, \beta_4. \tag{1.2}$$

Hence, when M and σ are bounded, it is possible to strengthen the Berry-Esseen theorem with an extension to higher dimensions.

In order to reflect the influence of $M(X_k)$ on average (similarly to β_3), rather than via the maximal value M, some refinements of the bounds (1.1)-(1.2) are given in Section 6.

For several classes of probability distributions, the functional $M(X)$ is of the order σ^{-d} when a random vector X has a covariance matrix $\sigma^d I_d$ (modulo d-dependent constants). Here is an example involving convexity properties of distributions.

Corollary 1.2. *Suppose that the random vectors X_k have log-concave densities, with mean zero and unit covariance matrix. Then*

$$\Delta_n \leq \frac{C_d}{\sqrt{n}} \tag{1.3}$$

with some constant C_d depending on d only. If additionally the distributions of X_k's are symmetric, then

$$\Delta_n \leq \frac{C_d}{n}. \tag{1.4}$$

Based on the application of the Fourier transform (which is typical in the study of various variants of the central limit theorems), the main arguments used in the proof of Theorem 1.1 employ the subbaditivity property of the maximum-of-density functional $M(X)$ with respect to convolutions. This tool has been introduced in the field of limit theorems only recently and is used in [10], [11]. Another ingredient in the proof involves the extension of the Statuljavičus separation theorem for characteristic functions to higher dimensions.

The paper is organized as follows. We start with remarks on isotropic constants and general bounds for $M(X)$ in terms of the covariance matrix of a random vector X (Section 2) and then discuss the question of how one can separate the absolute value of a given characteristic function $f(t)$ from 1 in terms of $M(X)$ outside a neighborhood of the origin (Sections 3-4). In Sections 5 we recall the subadditivity property of $M(X)$ and develop its applications to the integrability properties of powers of characteristic functions. Basic results on the normal approximation of products of characteristic functions are recalled in Section 6. In Section 7, we derive refined bounds on Δ_n, which are used in Section 8 to finish the proof of Theorem 1.1 and Corollary 1.2.

2. Lower Bounds on Maximum of Density via Covariance Matrix

To start with, first let us recall the general relation

$$M^2\sigma^2 \geq \frac{1}{12}, \tag{2.1}$$

which holds for any random variable X with standard deviation σ and maximum of density $M = M(X)$. As an early reference one can mention the paper by Statuljavičus [21], p. 651, where (2.1) is stated without derivation as an obvious fact. In the muldimensional situation, (2.1) is extended in the form

$$M^{2/d}\sigma^2 \geq \frac{1}{d+2}\,\omega_d^{-2/d}, \tag{2.2}$$

assuming that the random vector X has a covariance matrix $\sigma^2 I_d$ (such distributions are called isotropic). Here an equality is attained for the uniform distribution on every Euclidean ball in $\mathbb{R}^d$ and, in particular, for the unit ball $B_2 = \{x \in \mathbb{R}^d : |x| \leq 1\}$ with volume

$$\omega_d = \mathrm{vol}_d(B_2) = \frac{\pi^{d/2}}{\Gamma(\frac{d}{2}+1)}. \tag{2.3}$$

This extremal property of balls has been investigated in Convex Geometry in the context of bounding volume of slices of convex bodies, see Hensley [17] and Ball [3]. More precisely, it was shown in [3], Lemma 6, that, if the density of X satisfies $p(x) \leq p(0)$ for all $x \in \mathbb{R}^d$, then

$$p(0)^{2/d} \int_{\mathbb{R}^d} |x|^2\,dx \geq \frac{d}{d+2}\,\omega_d^{-2/d}.$$

This amounts to (2.2) in the case where X has mean zero, covariance matrix $\sigma^2 I_d$, and assuming that $p(x)$ is maximized at the origin. In [12], Proposition III.1, this inequality was generalized and strengthened as the property that, for any non-decreasing function $H = H(t)$ in $t \geq 0$, the moment-type functional

$$\int_{\mathbb{R}^d} H(M^{1/d}\,|x|)\,p(x)\,dx$$

is minimized for the uniform distribution on every Euclidean ball with center at the origin. For $H(t) = t^2$, this property implies (2.2), assuming that X has mean zero and covariance matrix $\sigma^2 I_d$. But the inequality (2.2) is translation invariant, so it continues to hold without the mean zero assumption.

The quantity $M^{1/d}\sigma$ is often called an isotropic constant of the distribution of X. It is also well-known that the relation (2.2) is in essense dimension-free, since it implies

$$M^{2/d}\sigma^2 \geq \frac{1}{2\pi e} \tag{2.4}$$

with an optimal constant on the right-hand side (attainable asymptotically for growing dimension d). More generally, if a random vector X has covariance matrix R, then

$$\big(M^2 \det(R)\big)^{1/d} \geq \frac{1}{2\pi e}.$$

This follows from the previous lower bound (2.4) by applying it to the random vector $Y = R^{-1/2}X$. It is isotropic with $\sigma(Y) = 1$ and $M(Y) = \sqrt{\det(R)}\, M(X)$.

In this connection, let us mention that in the class of isotropic distributions on $\mathbb{R}^d$ having log-concave densities, all these lower bounds can be reversed as

$$M^{2/d}\sigma^2 \leq K_d^2 \tag{2.5}$$

in terms of the maximal isotropic constant over this class for a fixed dimension d. In the equivalent form, the hyperplane conjecture raised in 1980's by Bourgain, which is still open, asserts that K_d is bounded by an absolute constant. This is true, for example, when additionally the density of X is symmetric about all coordinate axes (cf. e.g. [13]). As for the general log-concave case, at this moment the best result in this direction belongs to Klartag [18] with his bound $K_d^2 \leq C\log(d+1)$ for some absolute constant C. Let us refer an interested reader to [15] and [1] for the history of the problem and related results.

3. Separation of Characteristic Functions (Statuljavičus Theorem)

In order to apply the Fourier methods for the derivation of density bounds in a quantitative way, one has to realize how to separate the characteristic function

$$f(t) = \mathbb{E}\, e^{itX}, \quad t \in \mathbb{R},$$

of a random variable X from 1 outside a neighborhood of the origin (which is potentially possible due to the Riemeann-Lebesgue lemma). That is, the task is to derive estimates like $\sup_{|t|\geq t_0} |f(t)| < 1 - \delta$ with an arbitrary $t_0 > 0$ and some positive $\delta = \delta(t_0)$. An important step in this direction was made by Statuljavičus [21] who derived an upper bound which we prefer to state in an equivalent form.

Proposition 3.1. *Given a random variable X with standard deviation σ and finite maximum of density $M = M(X)$, its characteristic function satisfies*

$$|f(t)| \leq 1 - \frac{c}{M^2\sigma^2}\, \min\{\sigma^2|t|^2, 1\}, \quad t \in \mathbb{R}, \tag{3.1}$$

with some absolute constant $c > 0$.

Some generalizations of this result to the general case of unbounded densities and without moment assumptions are discussed in [8]. In fact, Statulevičius considered more complicated quantities for upper bounds reflecting the behavior of the density p of X on non-overlapping intervals of the real line and obtained the bound

$$|f(t)| \le \exp\Big\{-\frac{t^2}{96\,M^2\,(2\sigma|t|+\pi)^2}\Big\}$$

as a partial case of a more general relation. Therefore, let us include a shorter simplified argument aimed at the bound (3.1) only (without polishing the constant c).

Proof. In a more flexible form, the inequality (2.1) can be equivalently written as

$$\big(\operatorname{ess\,sup}_x q(x)\big)^2 \int_{-\infty}^{\infty} x^2 q(x)\,dx \ge \frac{1}{12}\,\Big(\int_{-\infty}^{\infty} q(x)\,dx\Big)^3, \tag{3.2}$$

holding true for any non-negative Borel measurable function $q(x)$ on the real line.

Consider the symmetrized random variable $\widetilde X = X - X'$, where X' is an independent copy of X. It has a positive characteristic function $|f(t)|^2$ and density

$$w(x) = \int_{-\infty}^{\infty} p(x+y)p(y)\,dy \le M.$$

Write

$$1-|f(2\pi t)|^2 = \int_{-\infty}^{\infty} (1-\cos(2\pi tx)\,w(x)\,dx = 2\int_{-\infty}^{\infty}\sin^2(\pi tx)\,w(x)\,dx.$$

In order to bound the last integral from below, one may use the elementary inequality

$$|\sin(\pi x)| \ge \|x\| = \min\{|x-k| : k\in\mathbb{Z}\},$$

where both sides represent 1-periodic functions. Assuming without loss of generality that $t>0$ and using $1-|f(s)|^2 \le 2\,(1-|f(s)|)$, this gives

$$1-|f(2\pi t)| \ge 4\int_W \|tx\|^2\,w(x)\,dx, \tag{3.3}$$

where we restrict the integration to the set $W = \{x\in\mathbb{R} : t|x| < N+\frac{1}{2}\}$ for a suitable integer $N \ge 0$. Let us split the integral in (3.3) into the sets

$$W_k = \Big\{x\in\mathbb{R} : k-\frac{1}{2} < t|x| < k+\frac{1}{2}\Big\}$$

and rewrite (3.3) as

$$\begin{aligned} 1-|f(2\pi t)| &\ge 4\sum_{k=-N}^{N}\int_{W_k} |tx-k|^2\,w(x)\,dx \\ &= 4t^2\sum_{k=-N}^{N}\int_{-\frac{1}{2t}}^{\frac{1}{2t}} y^2\,w\Big(y+\frac{k}{t}\Big)\,dy. \end{aligned}$$

Applying (3.2) to the functions $q_k(y) = w(y+\frac{k}{t})\,1_{[-\frac{1}{2t},\frac{1}{2t}]}(y)$ and using $w \le M$, we have

$$\int_{-\frac{1}{2t}}^{\frac{1}{2t}} y^2 w\Big(y+\frac{k}{t}\Big)\,dy \ge \frac{1}{12M^2}\,\Big[\int_{-\frac{1}{2t}}^{\frac{1}{2t}} w\Big(y+\frac{k}{t}\Big)\,dy\Big]^3 = \frac{1}{12\,M^2}\,\Big(\int_{W_k} w(x)\,dx\Big)^3.$$

Hence

$$1 - |f(2\pi t)| \geq \frac{t^2}{3M^2} \sum_{k=-N}^{N} Q_k^3, \quad \text{where} \;\; Q_k = \int_{W_k} w(x)\,dx.$$

Subject to $\sum_{k=-N}^{N} Q_k = Q$, the sum $\sum_{k=-N}^{N} Q_k^3$ is minimized when $Q_k = Q/(2N+1)$. This leads to

$$1 - |f(2\pi t)| \geq \frac{1}{3M^2}\, \frac{t^2}{(2N+1)^2}\, Q^3. \tag{3.4}$$

One should now maximize the right-hand side or choose a suitable N. Since $\mathbb{E}\widetilde{X} = 0$, $\mathbb{E}\widetilde{X}^2 = 2\sigma^2$, we get, by Chebyshev's inequality,

$$1 - Q = \mathbb{P}\Big\{|\widetilde{X}| \geq \frac{N + \frac{1}{2}}{t}\Big\} \leq 2\left(\frac{\sigma t}{N+\frac{1}{2}}\right)^2. \tag{3.5}$$

If $\sigma t > \frac{1}{4}$, we choose $N = [2\sigma t + \frac{1}{2}]$, in which case $Q \geq \frac{1}{2}$ and $2N + 1 \leq 12\,\sigma t$. Hence

$$\frac{t^2}{(2N+1)^2}\, Q^3 \geq \frac{t^2}{(12\,\sigma t)^2 \cdot 8} = \frac{1}{1152\,\sigma^2}$$

and thus

$$1 - |f(2\pi t)| \geq \frac{1}{3456\, M^2\sigma^2}, \quad \sigma t \geq 1/4.$$

If $\sigma t \leq \frac{1}{4}$, the choice $N = 0$ in (3.5) yields $Q \geq \frac{1}{2}$. By (3.4),

$$1 - |f(2\pi t)| \geq \frac{t^2}{24\, M^2}, \quad \sigma t \leq \frac{1}{4}.$$

□

4. Separation of Characteristic Functions (Reduction to Dimension One)

In the multidimensional case, the characteristic function

$$f(t) = \mathbb{E}\, e^{i\langle t, X\rangle}, \quad t \in \mathbb{R}^d,$$

admits a similar bound. The next statement is a preliminary step in the proof of Theorem 1.1.

Proposition 4.1. *Given a random vector X in $\mathbb{R}^d$ with covariance matrix $\sigma^2 I_d$, $\sigma > 0$, and a finite maximum of density $M = M(X)$, its characteristic function satisfies*

$$|f(t)| \leq 1 - \frac{c^d}{M^2\sigma^{2d}}\, \min\{\sigma^2|t|^2, 1\}, \quad t \in \mathbb{R}^d, \tag{4.1}$$

with some absolute constant $c > 0$.

Note that the isotropic constant $M^{1/d}\sigma$ is present on the right-hand side, which is bounded away from zero according to (2.4). For isotropic log-concave distributions, it is also bounded from above by a d-dependent constant according to (2.5). But in this case one can certainly obtain better bounds with decay of $f(t)$ at infinity.

For the proof, it looks natural to apply the one dimensional result to random variables $X_\theta = \langle \theta, X\rangle$ with unit vectors θ. However, it may happen that X_θ will have an unbounded density, which means that Proposition 3.1 is not applicable.

For example, in dimension $d = 2$, suppose that $X = (X_1, X_2)$ has a uniform distribution on the unbounded region $R = \{(x_1, x_2) : |x_1| \le \exp(-c|x_2|)\}$, $c > 0$, that is, with density

$$p(x_1, x_2) = \frac{c}{4}\, 1_R(x_1, x_2), \quad x_1, x_2 \in \mathbb{R}.$$

Then X_1 takes values in $(-1, 1)$ and has density

$$p_1(x_1) = \int_{-\infty}^{\infty} p(x_1, x_2)\, dx_2 = \frac{1}{2}\, \log \frac{1}{|x_1|}, \quad -1 < x_1 < 1,$$

which is unbounded near zero. It is easy to check that the random vector X is isotropic, that is, $\mathbb{E}X_1 X_2 = 0$ and $\mathbb{E}X_1^2 = \mathbb{E}X_2^2$ for a suitable constant c.

Proof of Proposition 4.1. The above example shows that a preliminary density truncation is desirable. Introduce a random vector X_r in $\mathbb{R}^d$ with parameter $r > 0$ with density

$$p_r(x) = \frac{1}{b_r}\, p_r(x)\, 1_{\{|x|<r\}}, \quad x \in \mathbb{R}^d,$$

where $b_r = \mathbb{P}\{|X| < r\}$ is a normalizing constant. Assuming that $d \ge 2$, we choose $r = \sigma\sqrt{2d}$, which guarantees, by Markov's inequality, that

$$b_r = 1 - \mathbb{P}\{|X| \ge r\} \;\ge\; 1 - \frac{\mathbb{E}\,|X|^2}{r^2} \;=\; 1 - \frac{d\sigma^2}{r^2} \;=\; \frac{1}{2}.$$

Put $t = s\theta$, $s \in \mathbb{R}$, $|\theta| = 1$, and consider the characteristic functions

$$g_r(s) = \mathbb{E}\, e^{i\langle t, X_r\rangle} = \mathbb{E}\, e^{is\langle \theta, X_r\rangle}, \quad g(s) = \mathbb{E}\, e^{i\langle t, X\rangle} = \mathbb{E}\, e^{is\langle \theta, X\rangle}.$$

By construction,

$$\begin{aligned}
1 - |g(s)|^2 &= 2\int_{\mathbb{R}^d} \sin^2\Big(\frac{\langle t, x\rangle}{2}\Big)\, p(x)\, dx \\
&\ge 2\int_{|x|<r} \sin^2\Big(\frac{\langle t, x\rangle}{2}\Big)\, p(x)\, dx \\
&= 2b_r \int_{\mathbb{R}^d} \sin^2\Big(\frac{\langle t, x\rangle}{2}\Big)\, p_r(x)\, dx \\
&\ge \int_{\mathbb{R}^d} \sin^2\Big(\frac{\langle t, x\rangle}{2}\Big)\, p_r(x)\, dx \;=\; \frac{1}{2}\, (1 - |g_r(s)|^2).
\end{aligned}$$

Thus,

$$1 - |g(s)|^2 \ge \frac{1}{2}\, (1 - |g_r(s)|^2). \tag{4.2}$$

In order to bound from below the right-hand side, first note that

$$\begin{aligned}
\mathrm{Var}(\langle \theta, X_r\rangle) &= \frac{1}{2b_r^2} \int_{|x|<r}\int_{|y|<r} \langle \theta, x - y\rangle^2\; p(x)p(y)\, dxdy \\
&\le \frac{1}{2b_r^2} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \langle \theta, x - y\rangle^2\; p(x)p(y)\, dxdy \;=\; \frac{\sigma^2}{b_r^2} \;\le\; 4\sigma^2.
\end{aligned}$$

Thus,

$$\mathrm{Var}(\langle\theta, X_r\rangle) \le 4\sigma^2. \tag{4.3}$$

The maximum of density of the random variable $\langle\theta, X_r\rangle$ can also be related to $M = M(X)$. For simplicity, let $\theta = e_1 = (1, 0, \dots, 0)$, so that $\langle\theta, X_r\rangle$ has the one dimensional density

$$\int_{\mathbb{R}^{d-1}} p_r(x,y)\,dy \le \frac{1}{b_r}\int_{|y|<r} p(x,y)\,dy \ \le 2\,\omega_{d-1} r^{d-1} M, \quad x \in \mathbb{R}.$$

Thus, for any unit vector θ in $\mathbb{R}^d$,

$$M(\langle\theta, X_r\rangle) \le 2\omega_{d-1} r^{d-1} M. \tag{4.4}$$

Therefore, by Proposition 3.1 applied to the characteristic function $g_r(s)$ with its properties (4.3)-(4.4), it follows that

$$1 - |g_r(s)|^2 \ge \frac{c}{\omega_{d-1}^2 r^{2(d-1)} M^2 \sigma^2}\,\min\{\sigma^2 s^2, 1\}$$

up to some absolute constant $c > 0$. Using this in (4.2), we arrive at the similar relation

$$1 - |g(s)|^2 \ge \frac{c}{\omega_{d-1}^2 r^{2(d-1)} M^2 \sigma^2}\,\min\{\sigma^2 s^2, 1\},$$

that is,

$$1 - |g(s)|^2 \ge \frac{c}{\omega_{d-1}^2 (2d)^{d-1} M^2 \sigma^{2d}}\,\min\{\sigma^2 s^2, 1\}.$$

To simplify the constants, recall the formula (2.3) and Batir's bounds for the Gamma function ([4])

$$\sqrt{2e}\,\Big(\frac{x}{e}\Big)^x \le \Gamma\Big(x + \frac{1}{2}\Big) \le \sqrt{2\pi}\,\Big(\frac{x}{e}\Big)^x, \quad x \ge \frac{1}{2}.$$

The lower bound gives

$$\omega_{d-1}(2d)^{\frac{d-1}{2}} = \frac{\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d+1}{2})}\,(2d)^{\frac{d-1}{2}} \le \frac{1}{2\sqrt{\pi e d}}\,(4\pi e)^{d/2}.$$

It remains to note that $1 - |g(s)|^2 \le 2\,(1 - |g(s)|)$, and (4.1) follows. □

5. **Maximum of Convolved Densities**

Convolved densities are known to have improved smoothing properties. First, let us emphasize the following general fact.

Proposition 5.1. *If independent random vectors $X_1, \dots, X_m$ $(m \ge 2)$ with values in $\mathbb{R}^d$ have bounded densities, then the sum $S_m = X_1 + \dots + X_m$ has a bounded uniformly continuous density vanishing at infinity.*

Proof. Denote by q_k the densities of X_k and assume that $q_k(x) \le M_k$ for all $x \in \mathbb{R}^d$ with some constants M_k $(k \le m)$. By the Plancherel theorem, for the characteristic functions

$v_k(t) = \mathbb{E}\, e^{i\langle t, X_k\rangle}$, we have

$$\begin{aligned}\int_{\mathbb{R}^d} |v_k(t)|^m\, dt &\leq \int_{\mathbb{R}^d} |v_k(t)|^2\, dt \,=\, (2\pi)^d \int_{\mathbb{R}^d} q_k(x)^2\, dx \\ &\leq (2\pi)^d \int_{\mathbb{R}^d} M_k q_k(x)\, dx \,=\, (2\pi)^d M_k,\end{aligned}$$

where we used the property $|v_k(t)| \leq 1$, $t \in \mathbb{R}^d$. Hence, by Hölder's inequality, the characteristic function $f(t) = v_1(t) \dots v_m(t)$ of S_m is integrable and has L^1-norm

$$\begin{aligned}\int_{\mathbb{R}^d} |f(t)|\, dt &\leq \Big(\int_{\mathbb{R}^d} |v_1(t)|^m\, dt\Big)^{1/m} \dots \Big(\int_{\mathbb{R}^d} |v_m(t)|^m\, dt\Big)^{1/m} \\ &\leq (2\pi)^d\, (M_1 \dots M_m)^{1/m} \,<\, \infty.\end{aligned} \tag{5.1}$$

One may conclude that the random variable S_m has a bounded, uniformly continuous density expressed by the inversion Fourier formula

$$q(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\langle t,x\rangle} f(t)\, dt, \quad x \in \mathbb{R}. \tag{5.2}$$

Since f is integrable, it also follows that $q(x) \to 0$ as $|x| \to \infty$, by the Riemann-Lebesgue lemma. □

Since, by (5.2),

$$q(x) \leq \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |f(t)|\, dt$$

for all $x \in \mathbb{R}$, the inequality (5.1) also implies that

$$M(S_m) \leq (M(X_1) \dots M(X_m))^{1/m}. \tag{5.3}$$

However, the relation (5.3) does not correctly reflect the behavior of $M(S_m)$ with respect to the growing parameter m, especially in the i.i.d. situation. A more precise statement from [7] is described in the following relation, where the geometric mean of maxima is replaced with the harmonic mean.

Proposition 5.2. *Given independent random vectors X_k, $1 \leq k \leq m$, with values in $\mathbb{R}^d$, one has*

$$\frac{1}{M(S_m)^{2/d}} \geq \frac{1}{e} \sum_{k=1}^m \frac{1}{M(X_k)^{2/d}}. \tag{5.4}$$

This bound may be viewed as a counterpart of the entropy power inequality in Information Theory. It is derived by applying the Hausdorff-Young inequality with best constants (due to Beckner and Lieb). The constant $1/e$ is optimal and is attained asymptotically as $d \to \infty$ for random vectors unifomly distributed on Euclidean balls. However, in dimension $d = 1$, it can be improved to $1/2$, which follows from results due to Rogozin [20] and Ball [2].

One useful consequence of (5.4) is the following bound on the L^{2m}-norms of characteristic functions.

Proposition 5.3. *If $f(t)$ is the characteristic function of a random vector X in $\mathbb{R}^d$, then for any integer $m \geq 1$,*

$$\frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |f(t)|^{2m}\, dt \leq \Big(\frac{e}{2m}\Big)^{d/2} M(X). \tag{5.5}$$

Proof. We apply Proposition 5.2 to $2m$ summands $X_1, -X_1', \dots, X_m, -X_m'$, assuming that X_k, X_k' are independent copies of X. Introduce the symmetrized random vector $\widetilde S_m = S_m - S_m'$, where S_m' is an independent copy of S_m. By (4.4), we then get

$$M(\widetilde S_m) \leq \Big(\frac{e}{2m}\Big)^{d/2} M(X).$$

In addition, $\widetilde S_m$ has characteristic function $|f(t)|^{2m}$. If $M(X)$ is finite, one may apply Proposition 5.1 and conclude that $\tilde S_m$ has a bounded continuous density $q_m(x)$ which is vanishing at infinity. Moreover, $q_m(x)$ is maximized at $x = 0$, and its value at this point is described by the inversion formula (5.2) which gives

$$M(\widetilde S_m) = q_m(0) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |f(t)|^{2m}\, dt.$$

$\square$

Using (5.3), one can obtain a similar relation, but without the factor $(\frac{e}{2m})^{d/2}$ in (5.5).

When $M(X)$ is finite and m is large, the bound (5.5) may be considerably sharpened asymptotically with respect to m when restricting the integration to the regions $|t| \geq \varepsilon > 0$.

Proposition 5.4. *Let f be the characteristic function of a random variable X with covariance matrix $\sigma^2 I_d$ ($\sigma > 0$) and finite $M = M(X)$. For any $\varepsilon > 0$ and $n \geq 2$,*

$$\int_{|t| \geq \varepsilon} |f(t)|^n\, dt \leq \Big(\frac{8\pi^2 e}{n}\Big)^{d/2} M \exp\Big\{-\frac{c^d n}{M^2 \sigma^{2d}}\, \min(\sigma^2 \varepsilon^2, 1)\Big\} \tag{5.6}$$

with some absolute constant $c > 0$.

Proof. Since the random vector X has a density, we have $\delta_f(\varepsilon) = \max_{|t| \geq \varepsilon} |f(t)| < 1$, by the Riemann-Lebesgue lemma. Moreover, by Proposition 4.1,

$$\delta_f(\varepsilon) \leq 1 - \frac{c^d}{M^2 \sigma^{2d}}\, \min(\sigma^2 \varepsilon^2, 1) \tag{5.7}$$

with some absolute constant $c > 0$. If $n = 2$, we apply (5.5) with $m = 1$ and use the property that $M^2 \sigma^{2d}$ is bounded away from zero, cf. (2.4). In the case $n \geq 3$, write $n = 2m + k$ with $k = 1$ or $k = 2$ for $n \leq 5$ and with $m = [\frac{n}{3}]$, $k = n - 2m$ for $n \geq 6$. Then, by (5.5) and (5.7),

$$\begin{aligned} \int_{|t| \geq \varepsilon} |f(t)|^n\, dt &\leq \delta_f(\varepsilon)^k \int_{\mathbb{R}^d} |f(t)|^{2m}\, dt \\ &\leq \Big(\frac{2\pi^2 e}{m}\Big)^{d/2} M \exp\Big\{-\frac{c^d k}{M^2 \sigma^{2d}}\, \min(\sigma^2 \varepsilon^2, 1)\Big\}. \end{aligned}$$

It remains to note that $m \geq \frac{1}{4}\, n$ and $k \geq c_1 n$ for some absolute constant $c_1 > 0$. $\square$

6. Normal Approximation for Products of Characteristic Functions

Let us now recall standard results about the approximation of products of charactersitic functions. Consider the sum

$$S_n = \xi_1 + \cdots + \xi_n$$

of independent random variables ξ_k with mean zero and standard deviations b_k such that $b_1^2 + \cdots + b_n^2 = 1$, so that S_n has mean zero and variance one.

In terms of the characteristic functions $v_k(t) = \mathbb{E}\, e^{it\xi_k}$, the characteristic function of S_n represents the product

$$f_n(t) = \mathbb{E}\, e^{itS_n} = v_1(t) \dots v_n(t), \quad t \in \mathbb{R}.$$

Under higher order moment assumptions, it may be approximated by the standard normal characteristic function $g(t) = e^{-t^2/2}$ or its Edgeworth corrections on relatively large t-intervals by means of the Lyapunov coefficients

$$L_p = \sum_{k=1}^n \mathbb{E}\, |\xi_k|^p, \quad p > 2,$$

provided that they are small. We only mention such results for the particular indexes $p = 3$ and $p = 4$.

Proposition 6.1. *With some absolute constants $C > 0$ and $c > 0$,*

$$|f_n(t) - e^{-t^2/2}| \leq CL_3\, |t|^3\, e^{-ct^2}, \quad |t| \leq \frac{1}{L_3}. \tag{6.1}$$

Moreover, if $\mathbb{E}\xi_k^3 = 0$ for all $k \leq n$, then

$$|f_n(t) - e^{-t^2/2}| \leq CL_4\, t^4\, e^{-ct^2}, \quad |t| \leq \frac{1}{\sqrt{L_4}}. \tag{6.2}$$

The inequalities (6.1)-(6.2) are often stated in a slightly different form. For example, in Petrov ([19], p. 109), the relation (6.1) is derived on a smaller interval $|t| \leq \frac{1}{4L_3}$ with $C = 16$ and $c = 1/3$. As can be seen from the proof or properly modifying it, the interval of approximation can be increased to $|t| \leq \frac{c_0}{L_3}$ with some absolute constant $c_0 > 1$ at the expense of a smaller value of c and a larger value of C. Similar relations with arbitrary real $p > 2$ can be found in the review [6].

In the non-interesting case, where L_3 or L_4 are greater than 1, (6.1)-(6.2) hold true on the larger interval $|t| \leq 1$. This can be seen from the Taylor integral fromula for the function $h(t) = f_n(t) - e^{-t^2/2}$ about the point $t = 0$, using the property that the first 2 and 3 derivatives of $h(t)$ at the origin are respectively vanishing.

In general, the function $p \to L_p^{\frac{1}{p-2}}$ is non-decreasing with $L_p \geq n^{-\frac{p-2}{2}}$. In particular,

$$\frac{1}{\sqrt{n}} \leq L_3 \leq \sqrt{L_4}. \tag{6.3}$$

Since $\mathbb{E}\, |\xi_k|^p \geq (\mathbb{E}\, \xi_k^2)^{p/2}$, we also have

$$L_p \geq b_1^p + \cdots + b_n^p \geq (\max_k b_k)^p. \tag{6.4}$$

Now, if instead of S_n, one considers the normlized sum

$$Z_n = \frac{X_1 + \dots + X_n}{\sqrt{n}},$$

these results can be correspondingly reformulated. Then one should assume that the independent random variables X_k have mean zero and standard deviations σ_k such that $\sigma_1^2 + \dots + \sigma_n^2 = n$. In this setting, it is more natural to represent the Lyapunov coefficients as

$$L_p = n^{-\frac{p-2}{2}}\, \beta_p, \quad \beta_p = \frac{1}{n} \sum_{k=1}^n \mathbb{E}\,|X_k|^p.$$

Thus, $\beta_p = \mathbb{E}\,|X|^p$ when X_k are independent copies of a random variable X.

In general, from (6.3)-(6.4) it follows that

$$1 \leq \beta_3 \leq \sqrt{\beta_4}, \quad 1 \leq \max_k \sigma_k \leq (n\beta_p)^{1/p}. \tag{6.5}$$

Finally, let us state Proposition 6.1 once more about the normalized sums.

Proposition 6.2. *With some absolute constants $C > 0$ and $c > 0$, the characteristic function $f_n(t)$ of Z_n satisfies*

$$|f_n(t) - e^{-t^2/2}| \leq \frac{C\beta_3}{\sqrt{n}}\, |t|^3\, e^{-ct^2}, \quad |t| \leq \frac{\sqrt{n}}{\beta_3}. \tag{6.6}$$

Moreover, if $\mathbb{E}X_k^3 = 0$ for all $k \leq n$, then

$$|f_n(t) - e^{-t^2/2}| \leq \frac{C\beta_4}{n}\, t^4\, e^{-ct^2}, \quad |t| \leq \frac{\sqrt{n}}{\sqrt{\beta_4}}. \tag{6.7}$$

7. Refinement of Theorem 1.1

Let us return to the setting of Theorem 1.1 and assume that the independent random vectors X_k have mean zero, covariance matrix $\sigma_k^2 I_d$, and finite maxima of densities $M_k = M(X_k)$. Recall the notations

$$\beta_3 = \sup_{|\theta|=1} \bigg[\frac{1}{n} \sum_{k=1}^n \mathbb{E}\,|\,\langle \theta, X_k\rangle\,|^3\bigg], \quad \beta_4 = \sup_{|\theta|=1} \bigg[\frac{1}{n} \sum_{k=1}^n \mathbb{E}\,\langle \theta, X_k\rangle^4\bigg],$$

and

$$\Delta_n = \sup_x |p_n(x) - \varphi(x)|.$$

Here we prove this theorem in a somewhat more general form (although more complicated).

Theorem 7.1. *With some positive absolute constants C, c, the density p_n of Z_n satisfies*

$$\Delta_n \leq \frac{C^d \beta_3}{\sqrt{n}} + C^d\, (M_1 \dots M_n)^{\frac{1}{n}} \exp\Big\{-c^d \sum_{k=1}^n \frac{1}{M_k^2 \sigma_k^{2d}}\, \min(\sigma_k^2/\beta_3^2, 1)\Big\}. \tag{7.1}$$

Moreover, if $\mathbb{E}\,\langle \theta, X_k\rangle^3 = 0$ *for all* $\theta \in \mathbb{R}^d$ *and* $k \leq n$ *(in particular, if the distributions of* X_k *are symmetric), then*

$$\Delta_n \leq \frac{C^d \beta_4}{n} + C^d\,(M_1 \dots M_n)^{\frac{1}{n}} \exp\Big\{ - c^d \sum_{k=1}^n \frac{1}{M_k^2 \sigma_k^{2d}}\, \min(\sigma_k^2/\beta_4, 1)\Big\}. \tag{7.2}$$

Proof. Since necessarily $M \geq c^d$ and $\sqrt{\beta_4} \geq \beta_3 \geq 1$ (cf. (2.4)), the inequalities (7.1)-(7.2) are fulfilled automatically for $n = 1$. So, assume that $n \geq 2$.

In terms of the characteristic functions $v_k(t) = \mathbb{E}\, e^{i\langle t, X_k\rangle}$, the characteristic function of Z_n is given by the product

$$f_n(t) = \mathbb{E}\, e^{i\langle t, Z_n\rangle} = v_1\Big(\frac{t}{\sqrt{n}}\Big) \dots v_n\Big(\frac{t}{\sqrt{n}}\Big).$$

Applying Hölder's inequality and then Proposition 5.3, we get

$$\int_{\mathbb{R}^d} |f_n(t)|\, dt \leq \prod_{k=1}^n \bigg(\int_{\mathbb{R}^d} \Big|v_k\Big(\frac{t}{\sqrt{n}}\Big)\Big|^n\, dt\bigg)^{1/n} < \infty.$$

Hence, one may apply the Fourier inversion formula to represent the densities of Z_n as

$$p_n(x) = \frac{1}{2\pi} \int_{\mathbb{R}^d} e^{-i\langle t, x\rangle} f_n(t)\, dt, \quad x \in \mathbb{R}^d.$$

Using a similar representation for the standard normal density, we get

$$\Delta_n \leq \frac{1}{2\pi} \int_{\mathbb{R}^d} |f_n(t) - e^{-|t|^2/2}|\, dt. \tag{7.3}$$

Next, we split integration in (7.3) into the two regions. From Proposition 5.4 applied to the characteristic function $v_k(t)$ of X_k, we obtain that, for any $\varepsilon > 0$,

$$\int_{|t| \geq \varepsilon} |v_k(t)|^n\, dt \leq \Big(\frac{C_0}{\sqrt{n}}\Big)^d M_k \exp\Big\{ - c^d\, \frac{n}{M_k^2 \sigma_k^{2d}}\, \min(\sigma_k^2 \varepsilon^2, 1)\Big\},$$

where $C_0 = 2\pi\sqrt{2e}$. Equivalently,

$$\int_{|t| \geq \varepsilon\sqrt{n}} \Big|v_k\Big(\frac{t}{\sqrt{n}}\Big)\Big|^n\, dt \leq C_0^d\, M_k \exp\Big\{ - c^d\, \frac{n}{M_k^2 \sigma_k^{2d}}\, \min(\sigma_k^2 \varepsilon^2, 1)\Big\}.$$

Applying once more Hölder's inequality

$$\int_{|t| \geq \varepsilon\sqrt{n}} |f_n(t)|\, dt \leq \prod_{k=1}^n \bigg(\int_{|t| \geq \varepsilon\sqrt{n}} \Big|v_k\Big(\frac{t}{\sqrt{n}}\Big)\Big|^n\, dt\bigg)^{1/n},$$

we then get

$$\int_{|t| \geq \varepsilon\sqrt{n}} |f_n(t)|\, dt \ \leq\ C_0^d\, (M_1 \dots M_n)^{\frac{1}{n}} \exp\Big\{ - c^d \sum_{k=1}^n \frac{\min(\sigma_k^2 \varepsilon^2, 1)}{M_k^2 \sigma_k^{2d}}\Big\}.$$

Since

$$\int_{|t| \geq \varepsilon\sqrt{n}} e^{-|t|^2/2}\, dt \ \leq\ C^d e^{-n\varepsilon^2/4}$$

for some absolute constant $C > 0$, it follows that

$$\int_{|t|\ge\varepsilon\sqrt{n}} \big|f_n(t) - e^{-|t|^2/2}\big|\,dt \ \le \ C^d e^{-n\varepsilon^2/4}$$

$$+ \, C^d\,(M_1 \dots M_n)^{\frac{1}{n}} \exp\Big\{-c^d \sum_{k=1}^n \frac{\min(\sigma_k^2\varepsilon^2,1)}{M_k^2\sigma_k^{2d}}\Big\}. \tag{7.4}$$

We now turn to the other region $|t| < \varepsilon\sqrt{n}$. By the mean zero and isotropy assumption on the distribution of X_k, we have $\mathbb{E}\,\langle\theta, X_k\rangle = 0$ and $\mathbb{E}\,\langle\theta, X_k\rangle^2 = 1$ for any unit vector θ in $\mathbb{R}^d$. Therefore, we are in position to apply Proposition 6.2 to the random variables $\langle\theta, X_k\rangle$. Write $t = s\theta$ for $s > 0$ and $|\theta| = 1$. Since $s \to f_n(s\theta)$ represents the characteristic function of the normalized sum of $\langle\theta, X_k\rangle$, it satisfies, by (6.6),

$$|f_n(s\theta) - e^{-s^2/2}| \le \frac{C\beta_3(\theta)}{\sqrt{n}}\, s^3\, e^{-cs^2}, \quad s \le \frac{\sqrt{n}}{\beta_3(\theta)}, \tag{7.5}$$

where

$$\beta_3(\theta) = \frac{1}{n}\sum_{k=1}^n \mathbb{E}\,|\,\langle\theta, X_k\rangle\,|^3.$$

In (7.5), $\beta_3(\theta)$ can be replaced with the larger value β_3, which leads to

$$|f_n(t) - e^{-|t|^2/2}| \le \frac{C\beta_3}{\sqrt{n}}\,|t|^3\,e^{-c|t|^2}, \quad |t| \le T_n = \frac{\sqrt{n}}{\beta_3}.$$

This readily yields

$$\int_{|t|\le T_n} |f_n(t) - e^{-|t|^2/2}|\,dt \le \frac{C^d\beta_3}{\sqrt{n}}.$$

with some absolute constant C. One can now combine this inequality with (7.4) by choosing $\varepsilon = \frac{1}{\beta_3}$. Since $\beta_3 \ge 1$, the resulting inequality in (7.3) yields (7.1).

In the second scenario, we similarly apply the inequality (6.7) and combine it once more with (7.4) by choosing $\varepsilon = \frac{1}{\sqrt{\beta_4}}$. □

8. **Proof of Theorem 1.1 and Corollary 1.2**

The right-hand sides in (7.1)-(7.2) may be bounded and simplified in terms of the functionals

$$M = \max_k M(X_k), \quad \sigma^2 = \max_k \sigma_k^2.$$

Proof of Theorem 1.1. The k-term in the sum in (7.1) represents a decreasing function with respect to σ_k^2 and M_k^2. Hence the second summand on the right-hand side in (7.1) does not exceed

$$C^d M \exp\Big\{-\frac{c^d n}{M^2\sigma^{2d}}\,\min(\sigma^2/\beta_3^2, 1)\Big\}.$$

Moreover, since $\sigma^2 \ge 1$ and $\beta_3 \ge 1$, necessarily $\min(\sigma^2/\beta_3^2, 1) \ge \beta_3^{-2}$. This further simplifies the latter expression to

$$C^d M \exp\Big\{-\frac{c^d n}{M^2\sigma^{2d}\beta_3^2}\Big\}.$$

Using $e^{-x} < x^{-1/2}$ $(x > 0)$, from (7.1) we obtain that

$$\Delta_n \le \frac{C^d \beta_3}{\sqrt{n}} + C^d\,\frac{M^2 \sigma^d \beta_3}{\sqrt{n}}.$$

Here, the first term on the right-hand side is dominated by the second one up to the multiple, so that the above estimate is simplified to (1.1).

With similar arguments, the second summand on the right-hand side in (7.2) does not exceed

$$C^d M \exp\Big\{-\frac{c^d n}{M^2 \sigma^{2d} \beta_4}\Big\}.$$

Using $e^{-x} < x^{-1}$ $(x > 0)$, from (7.2) we therefore obtain that

$$\Delta_n \le \frac{C^d \beta_4}{n} + C^d\,\frac{M^3 \sigma^{2d} \beta_4}{n}.$$

Again, the first term on the right-hand side is dominated by the second one up to the multiple, so that the above estimate is simplified to (1.2). This proves Theorem 1.1. □

Proof of Corollary 1.2. First we need to mention that, if a random vector X has a log-concave density, so does any linear functional $\xi = \langle \theta, X \rangle$. This is a consequence of a well-known characterization of log-concave measures due to Borell [14]. He also derived a large deviation bound for norms under log-concave measures, which implies in dimension one that L^p-norms of ξ are equivalent to each other. More precisely,

$$(\mathbb{E}\,|\xi|^p)^{1/p} \le Cp\,(\mathbb{E}\xi^2)^{1/2}$$

for all $p > 2$ with some absolute constant C. Applying this with $X = X_k$ in the case $\sigma_k = 1$, we conclude that β_3 and β_4 are bounded by absolute constants. Hence, the inequalities (1.1)-(1.2) are respectively simplified to

$$\Delta_n \le \frac{C^d M^2}{\sqrt{n}}, \quad \Delta_n \le \frac{C^d M^3}{n}.$$

To prove (1.3)-(1.4), it remains to recall the bound (2.5) which gives $M \le K_d^d$, where K_d is the maximal isotropic constant for the class of log-concave probability distributions on $\mathbb{R}^d$.

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